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斯坦纳圆链的求解

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摘 要 阐述在给定合理的允许误差的条件下,用CAGD程序证明斯坦纳圆链的命题,并给出工程方面的潜在应用实例.

关键词 圆,CAGD
中图法分类号 TP391

斯坦纳圆链 CAD

Study on Steiner Chain of Circles

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Abstract The thesis of Steiner Chain of Circles and the special features of the chain are proven by a CAGD program. Some potential applications in the engineering field are given.

Key words Circle, CAGD

1 INTRODUCTION

There is no clear border line between the CAGD, the CAD and the CAM in the field of computer application. To implement CAGD/CAM programs a first essential is that the accuracy of graphs and the machining error shall be considered[1].

At present, the distinct feature of graphs displaying in dynamic or static way is very important for studying geometric problems such as the Steiner Chain of Circles (see Fig. 1(b)), since this problem concerns somewhat "dynamically" drawing graphs and displaying them on screens.

Study of the relations between the circles and the degenerate circles (points or straight lines) has interested many mathematicians all over the world[2,3].

In engineering practice, for example, in designing an outershape of automobiles and profiling a blade of hydromachines or turbomachines, there are a lot of chances to construct circles being tangent to each other[4].

2 STATEMENT OF THE STEINER CHAIN OF CIRCLES

Let $\odot O$ be the external circle, $\odot O2$ the internal circle and e the distance between their centers or the eccentricity. The intermediate circles, $\odot O3f$, $\odot O3s$, $\odot O33$, $\odot O34, \cdots, \odot O3(N-1)$, $\odot O3N$, are described as touching these two fixed circles and one another.

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In general, as a result of drawing, the first circle $\odot O3f$ and the tail circle $\odot O3N$ are either intersected or separated (see Fig. 1(a), (c)). But under certain conditions, called "associated" or "coexistent", the two circles will be tangent and make up a closed chain. And then, this chain will keep closed forever no matter what the central angle of the first circle $\odot O3f$ is defined.

This geometric thesis has not been rigorously proven yet[5].

In other words, the Steiner Chain of Circles can be compared to a ball bearing, in which the balls are made of elastic material and with an eccentricity e between the outer and the inner rings. The working parts always keep contact in operation.

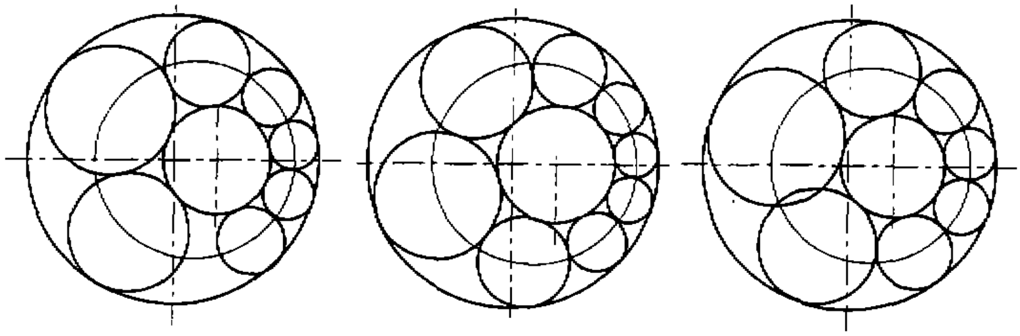


Fig. 1 The ending of the chain

3 THE GEOMETRIC MODEL AND ITS STANDARDIZATION

To study the problem of Steiner Chain of Circles, we are confronted with a multivariable function:

$$e = f(R, R2, R3i, N) \tag{1}$$

where e distance between centers or eccentricity,

R radius of the external circle,

$R2$ radius of the internal circle,

$R3i$ radii of the intermediate circles,

N number of the intermediate circles.

It is very hard to get an analytic solution for such a complex function.

As a result of scaling-up or scaling-down and system transformation, any of the two encirclement circles in a system of rectangular coordinates will get the final form (see Fig. 2), called "Standardized Model". The $\odot O$ can be understood as a unit circle.

In order to perform the numerical calculation, to display the results on screen and output on printer or plotter properly, and furthermore to compare the graphs with those drawn by hand, let $R=100$(mm).

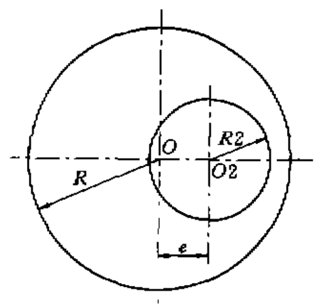
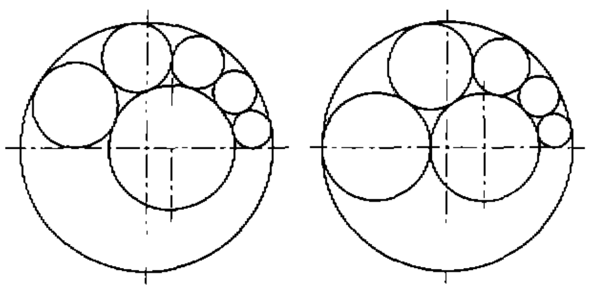


Fig. 2 The standardized Model

4 THE ESTABLISHMENT OF BASES OF GEOMETRIC GRAPHS TO BE VERIFIED AND THE CRITERION OF THE CLOSURE OF A CHAIN

4.1 The half chain

We can construct the first circle $\odot O3f$ which is tangent to the right x-axis and the next circles one by one in sequence only in I and II quadrants, i.e., a half chain instead of a whole chain. Under the optimized e and $R2$ "partners" conditions, the tail circle $\odot O3(N/2)$ of the half chain will be tangent to the left x-axis for even-numbered N (see Fig. 3(a)), and will lie on the x-axis without leaning to either side for odd-numbered N Fig. 3(b). We call it the basic graph.



(a) (b)

Fig. 3 The half chain

The calculation of the half chain is an equivalent estimation of the whole chain due to the symmetry of the graph (see the flow chare). This way reduces the amount of calculation getting "double results with half the effort".

Obviously, for an even N the radius $R3(N/2)$ equals the ordinate Y of the center of $\odot O3(N/2)$ and for an odd N, the ordinate Y equals zero.

4.2 The criterion of tangency

In accordance with the tangency theorem, we know that if the first and the tail circles are tangent then the distance between centers of the circles $\odot O3f$ and $\odot O3t$ shall absolutely equal the sum of the radii, thus:

$$\overline{O3fO3t} \equiv R3f + R3t \qquad (2)$$

or

$$\overline{O3fO3t} - (R3f + R3t) = 0 \qquad (3)$$

In engineering practice we use the symbol $\backsimeq$ "approximately equal to".

$$\overline{O3fO3t} \backsimeq R3f + R3t \qquad (4)$$

It means that an error ε can be used to judge the accuracy of the difference between the line segments $\overline{O3fO3t}$ and the sum of the radii.

$$\overline{O3fO3t} - (R3f + R3t) = \varepsilon \qquad (5)$$

During the calculation a permissible error $[\varepsilon]$ is given.

Certainly, before giving a value $[\varepsilon]$ it is necessary to consider the increase of the error accumulated with the another half chain in III and IV quadrants and the effect of scaling-up and scaling-down on the absolute error of the original graphs.

Hence, $[\varepsilon]$ can be selected as small as desired according to the requirement of applications.

In our practice we select $[\varepsilon] = 0.05$, because the minimum width of line drawn by jet printer is equal to 0.08(mm).

5 DERIVATION OF THE ALGEBRAIC-GEOMETRIC EQUATIONS

From Fig. 4 we see that the circle $\odot O3f$ is tangent to the external and the internal fixed circles. Thus the ordered pairs of the coordinates of the sought for center (X, Y) can be found by solving the following simultaneous equations:

$$X^2 + Y^2 = (R - R3f)^2 \qquad (6)$$

$$(X - e)^2 + (Y - O)^2 = (R2 + R3f)^2 \qquad (7)$$

where R radius of the external circle,

e eccentricity or the distance between centers,

$R2$ radius of the internal circle,

$R3f$ radius of the first intermediate circle.

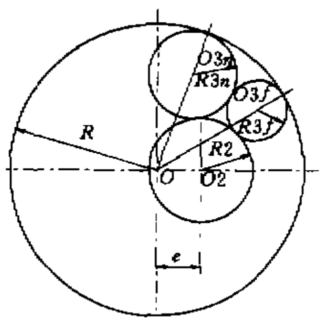


Fig. 4 The circles being tangent

To solve the simultaneous equations, (6) and (7), fundamental algebraic operations should be performed but nothing more than simplification of the equations by multiplying, eliminating terms, combining similar terms, etc. (The operations are omitted here). As a result, we can get the value of abscissa X of the $\odot O3f$ center:

$$X = -\frac{(R + R2)}{e} R3f + \left[\frac{(R^2 - R2^2)}{2e} + \frac{e}{2}\right] \qquad (8)$$

Let $X = K1\ R3f + K2$ (9)

where

$$K1 = -(\frac{R + R2}{e}) \quad (10)$$

$$K2 = \frac{R^2 - R2^2}{2e} + \frac{e}{2} \quad (11)$$

To find the ordinate Y of the $\odot O3f$ center we must go back to one of the original equations (6) or (7) and substitute the value X obtained from (9) for X.

On the analogy of this we can write the following simultaneous equations to evaluate the coordinates of the center of the next circle $\odot O3n$.

$$X^2 + Y^2 = (R - R3n)^2 \quad (12)$$

$$(X - X3f)^2 + (Y - Y3f)^2 = (R3f + R3n)^2 \quad (13)$$

$$X = K1\, R3n + K2 \quad (14)$$

In the similar way, solving the simultaneous equations we can get the abscissa Y of the next circle $\odot O3n$:

$$Y = -[\frac{R + R3f}{Y3f} + \frac{X3f}{Y3f}K1]R3n + [\frac{R^2 - R3f^2}{2Y3f} + \frac{X3f^2 + Y3f^2}{2Y3f} - \frac{X3f}{Y3f}K2] \quad (15)$$

For the sake of conciseness let

$$Y = K3\, R3n + K4 \quad (16)$$

where

$$K3 = -[\frac{R + R3f}{Y3f} + \frac{X3f}{Y3f}K1] \quad (17)$$

$$K4 = [\frac{R^2 - R3f^2}{2Y3f} + \frac{X3f^2 + Y3f^2}{2Y3f} - \frac{X3f}{Y3f}K2] \quad (18)$$

Substituting X and Y in (12) by the values obtained from (14), (16) and performing corresponding operations, we can obtain the following equation:

$$(K1^2+K3^2-1)R3n^2+2(K1K2+K3K4+R)R3n+(K2^2+K4^2-R^2)=0 \quad (19)$$

The formula (19) can be written in a simple form.

Let $\quad a\, R3n^2+b\, R3n+c=0 \quad (20)$

where

$$a= K_1^2 + K_3^2 - 1 \quad (21)$$

$$b= 2(K1K2 + K3K4 + R) \quad (22)$$

$$c= K2^2 + K4^2 - R^2 \quad (23)$$

Here we see obviously that (20) is a standard quadratic equation with one unknown. Therefore the general solution of (20), the roots of the equation, takes the following forms:

$$R3n1= \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad (24)$$

$$R3n2= \frac{-b - \sqrt{b^2 - 4ac}}{2a} \quad (25)$$

The roots $R3n1$ and $R3n2$ represent respectively the radius of the preceding and the rear circles, or called the next adjacent circles, which are tangent to the first circle $\odot O3f$. Substituting $R3n$ in (14) and (16) by $R3n1$ or $R3n2$ obtained from (24) or (25), we can find the corresponding ordered pairs (X,Y) of the abscissa and the ordinate of the $\odot O3n$ center.

6 PROOF OF THE EXISTENCE OF THE CLOSED CHAIN AND DATA PROCESSING

6.1 The examples are applied to prove numerically

These examples the existence of a closed Steiner chain of circles within certain [ε] given([ε] = 0.05). The central angles of the first circle $\odot O3f$ are arbitrary. We can see that all of the errors calculated are much less than the [ε] given (see Table1).

Table 1 Error Between Distance of Centers & Sum of Radii (The Central Angles of the First Circle are selected at will)

e=9.30576 $R2$=12.5 N=6		
No	Central Angle of the First Circle	Error Calculated
1	30°	9.15523 E-05
2	45°	9.536743 E-06
3	60°	6.0484985 E-04
4	120°	1.487732 E-04
5	195°	4.768372 E-04
6	300°	2.67028 E-05
e=25.0 $R2$=19.83646 N=12		
No	Central Angle of the First Circle	Error Calculated
1	3°10′22″	8.678436 E-5
2	15°30′	5.00679 E-04
3	50°7′	1.716614 E-04
4	145°3′	2.403259 E-04
5	205°1′	2.635956 E-04
6	295°4′	3.013611 E-04

The curves in Fig. 5 show the relation $R2=f(N)$, in which the combination of integer N and $R2$ makes up a closed chain.

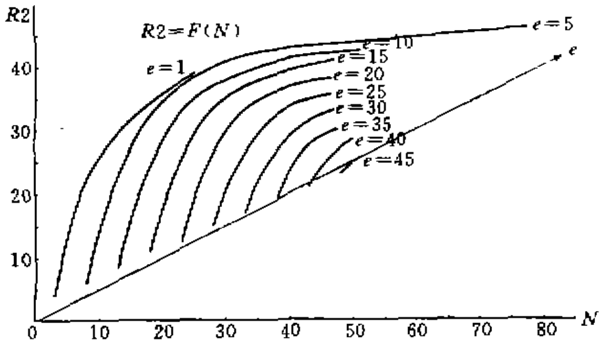


Fig. 5 The curve fitting

6.2 Two special cases

(1) The fixed circles ⊙O and ⊙$O2$ are tangent internally.

At the point of tangency the chain is linked by a degenerate circle, i.e., a zero circle ⊙$O3$ (see Fig. 6).

(2) The fixed circles ⊙O and ⊙$O2$ are concentric (see Fig. 7(a)).

The curves in FIg. 7(b) show the relation $R2=f(N)$ and $R3=g(N)$. Thus all combination of integer N and $R2$ makes up a closed chain.

Finally, the curves in Fig. 5 and Fig. 7(b) are established by fitting curve.

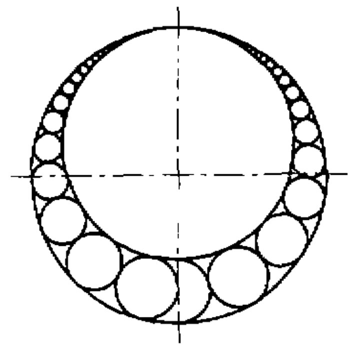
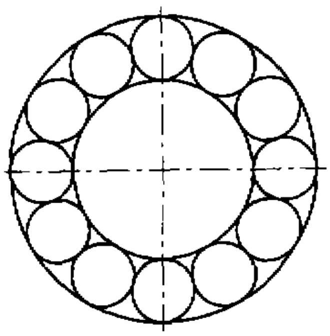
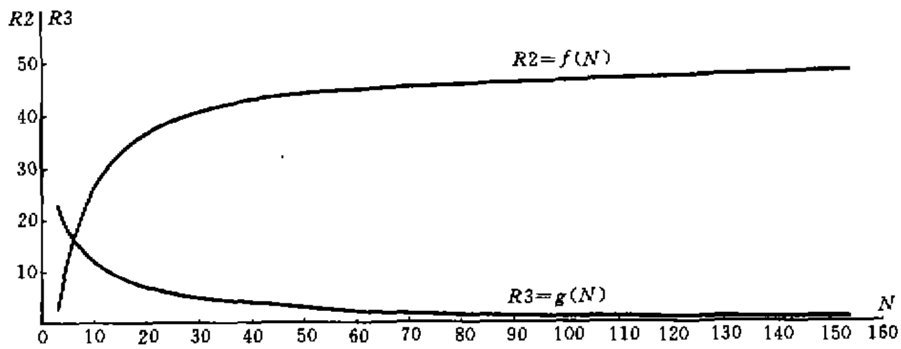


Fig. 6 Two circles being tangent internally



(a) The concentric circles



(b) The curve fitting

Fig. 7

7 CONCLUSION

(1) The solution of the problem is based upon giving an [ε] and getting an error to satisfy it through numerical computation. It's an example of the application of CAGD.

Similarly, an ordered pair (X,Y) represents a unique point in a rectangular coordinate system. For a given [ε] there is an "optimized ordered pair of partners" $(e, R2)$, which makes up a closed chain.

(2) The number of the ordered pairs $(e,R2)$ is infinite.

(3) It has to be said that it is a "relative" approximative solution, since π is used. It is known that π itself is a transcendental number. At present, by means of computers we are able to obtain the π value to millions of accurate decimal places. It can be found that the calculation using the different end-off and round-off π values will lead to different results, although the difference may be very small. So this kind of calculation will be always approximate and boundless or infinite.

8 POTENTIAL APPLICATIONS

(1) Display in advertisement.

(2) It's an ideal gas mixer in special use due to the splendid velocity distribution field where every point has a unique radius and rim speed.

(3) It can be used for special space arrangement of machine transmission. For example, in the textile equipment, every intermediate circle ⊙O3 can be designed as an independent output end, where a time-dependent angle speed is needed to get varied twisting density of the yarn and to make cloth with a beautiful decorative pattern, because the same material with different densities has different reflectivities of light.

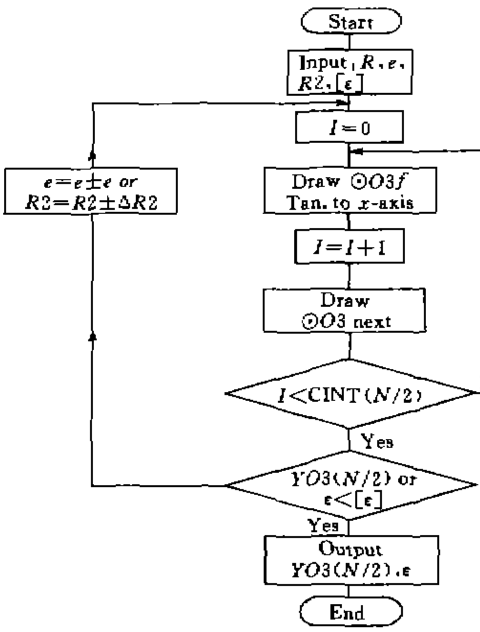


Fig. 8 Flow chart

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